THE DEPENDENCE OF FOCAL POINTS UPON CURVATURE FOR PROBLEMS OF THE CALCULUS OF VARIATIONS IN SPACE*

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In the Calculus of Variations, if an arc C_{01} which joins a space curve L and a fixed point 1 minimizes the integral

$$J = \int_0^1 f(x, y, y', z, z')\, dx \tag{1}$$

with respect to other curves joining L with 1, there will in general be a focal point 2 lying beyond 1 on the curve C of which C_{01} is a part, at which the minimizing property ceases. For the space problem it is well known that the minimizing arc C_{01} must an extremal and must be cut by L transversally at their point of intersection 0.† If these conditions are satisfied, C_{01} can be imbedded in a two-parameter family of extremals to each of which L is transversal and which will have an enveloping surface. If the enveloping surface has no singular point at its contact point with C, a further necessary condition for C_{01} to be a minimizing arc is that this contact point, which is the focal point mentioned above, does not lie between 0 and 1.

These results can be derived with the help of geometrical considerations, but the geometrical methods fail in case the enveloping surface has a singular point at its contact with C_{01}. It is the purpose of the present paper to derive the properties of the focal point by means of the second variation, and to show that they persist even when the enveloping surface may have a singularity. Further, the dependence of the position of the focal point upon the curvature of L will be discussed, and a result derived which is analogous to that which has already been found for the corresponding problem in the plane.‡ It is found that if a direction p, q, r through the intersection point 0 is properly chosen, and if π is the length of the segment in this direction which projects orthogonally into the radius of curvature of L at 0, then the distance from 0

* Presented to the Society (St. Louis), December 2, 1911.

† Mason and Bliss, *The properties of curves in space which minimize a definite integral*, Transactions of the American Mathematical Society, vol. 9 (1908), p. 440.

‡ Bliss, *The second variation of a definite integral when one end point is variable*, Transactions of the American Mathematical Society, vol. 3 (1902), p. 132.

to the focal point 2 along the extremal C varies monotonically with π. These results will then be applied to the case where the extremal is to minimize the integral with respect to curves joining a surface with a fixed point. The focal point for a surface on an extremal to which the surface is transversal at a point 0 is the nearest of the focal points of the normal sections of the surface through the point C.

§ 1. *Preliminary considerations.*

In this section a number of results will be stated which have been proved by previous writers and which are fundamental for what follows. In the integral

$$J = \int_0^1 f(x, y, y', z, z')\, dx \tag{1}$$

the function f will be assumed of class C'''* for all values (x, y, z, x', y', z') in a certain neighborhood of those corresponding to the arc C_{01} which is to be considered. Along this arc the problem is assumed to be regular, i. e.,

$$f_{y'y'}f_{z'z'} - f^2_{y'z'} \neq 0. \tag{2}$$

Further, the value of f on C at the point 0 where L and C intersect is not zero.

The equations of the fixed curve L are

$$x = x(u), \qquad y = y(u), \qquad z = z(u), \tag{3}$$

and the curve C, whose minimizing properties are to be discussed, will be represented by

$$y = \varphi(x), \qquad z = \psi(x). \tag{4}$$

Both L and C are supposed to be of class C''.

If C_{01} minimizes (1), it must be an extremal; that is, its equations (4) must be particular solutions of the Euler equations

$$f_y - f'_{y'} = 0, \qquad f_z - f'_{z'} = 0. \tag{5}$$

A second necessary condition is that C be cut by L transversally; in other words, the condition

$$px_u + qy_u + rz_u |^0 = 0 \tag{6}$$

must be satisfied at the point 0, where

$$p = f - \varphi' f_{y'} - \psi' f_{z'}, \qquad q = f_{y'} \qquad r = f_{z'}. \tag{7}$$

In equation (6) the arguments (x, y, y', z, z') of p, q, r are those belonging to the curve C at 0. A third necessary condition for a minimum is that the

* For the definition of the class of a curve see BOLZA, *Vorlesungen über Variationsrechnung*, p. 13.

quadratic form in η, ζ,

$$f_{y'y'}\eta^2 + 2f_{y'z'}\eta\zeta + f_{z'z'}\zeta^2, \tag{8}$$

be positive definite at each point of C_{01}. This is equivalent to the conditions,

$$f_{y'y'} > 0, \qquad f_{y'y'}f_{z'z'} - f^2_{y'z'} > 0 \tag{9}$$

along C_{01}, a form more analogous to Legendre's condition for the corresponding problem of the Calculus of Variations in the plane.

Suppose that C_{01} satisfies the preceding conditions. Since

$$f_{y'y'}f_{z'z'} - f^2_{y'z'} \neq 0$$

along the arc C_{01}, this arc can be imbedded in a two-parameter family of extremals,*

$$y = \varphi(x, u, v), \qquad z = \psi(x, u, v), \tag{10}$$

reducing to C_{01} for

$$u = 0, \qquad v = 0,$$

to each of which L is transversal. The functions φ, φ', ψ, ψ' will be of class C' for values of x, u, v, in a certain neighborhood of those defining the arc C_{01}. Along the curve L we have

$$y(u) = \varphi[x(u), u, v], \qquad z(u) = \psi[x(u), u, v], \tag{11}$$

and also the identity

$$px_u + qy_u + rz_u \equiv 0,$$

expressing the fact that L is transversal to the extremals. In this last identity the arguments of p, q, r are φ, φ', ψ, ψ' with $x(u)$ in place of x as in (11).

From (11) it follows that φ_v and ψ_v are zero at 0, and therefore the determinant of the family (10),

$$\Delta(x, u, v) = \begin{vmatrix} \varphi_u & \psi_u \\ \varphi_v & \psi_v \end{vmatrix}, \tag{12}$$

is also zero at that point. It can be proved that if $\Delta(x, u, v)$ vanishes at no other point on C_{01} this arc will furnish at least a weak minimum for the integral J with respect to curves joining L with 1. If on the other hand

$$\Delta(x, u, v) = 0$$

at some other point 2 on C_{01}, and if one of the determinants, say the first, of the matrix

$$\begin{Vmatrix} \Delta_u & \Delta_v \\ \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{Vmatrix} \tag{13}$$

* MASON AND BLISS, loc. cit., p. 460.

is not zero at that point, then the equations

$$\Delta(x, u, v) = 0$$

and

$$y = \varphi(x, u, v)$$

can be solved for u and v as functions of x, y in the neighborhood of the point 2. On substituting these values in

$$z = \psi(x, u, v),$$

there results the equation of the enveloping surface of the extremals, to which, in particular, C_{01} is tangent at 2. *The point 2 is called the focal point of L on C_{01}, and is characterized by the fact that it is the first point after 0 for which*

$$\Delta(x, u, v) = 0.$$

The existence of the enveloping surface with an ordinary point at 2 is dependent upon the assumption just made with respect to the determinants of the matrix (13). With the help of further properties of this surface it is possible to prove that *if the point 2 lies between 0 and 1 on the arc C_{01}, then C_{01} can not furnish even a weak minimum for the integral J with respect to curves joining the fixed curve L with the fixed point 1.* The theorem is true without the assumption on the matrix, as will be shown in § 5 with the help of the second variation.

If C_{01} is to minimize J with respect to curves joining a surface S,

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v), \tag{14}$$

and a fixed point 1, the arc must satisfy necessary conditions similar to those in the case just stated. It must be an extremal cut by S transversally at 0, and along it the Legendre condition must hold. The transversality conditions have the form,

$$px_u + qy_u + rz_u|_0 = 0, \qquad px_v + qy_v + rz_v|_0 = 0, \tag{15}$$

where the arguments for p, q, r, defined in (7), are the values belonging to C_{01} at 0. The extremal C_{01} can again be imbedded in a two-parameter family of extremals,*

$$y = \varphi(x, u, v), \qquad z = \psi(x, u, v) \tag{16}$$

to which S is transversal, and the equation

$$\Delta(x, u, v) = 0$$

for this family is the equation from which the focal point 2 may be determined. The properties of the focal point with respect to necessary or sufficient con-

* MASON AND BLISS, loc. cit., p. 448.

ditions for the minimizing arc in this case are similar to those stated above for the case when one end-point may vary on a fixed curve.

Since the curves of either of the families (10) or (16) are solutions of the Euler equations (5), their derivatives φ_u, ψ_u, φ_v, ψ_v are solutions of the Jacobi equations,

$$(17)\qquad \begin{aligned} &f_{yy}\eta + f_{yy'}\eta' + f_{yz}\zeta + f_{yz'}\zeta' - \frac{d}{dx}(f_{y'y}\eta + f_{y'y'}\eta' + f_{y'z}\zeta + f_{y'z'}\zeta') = 0, \\ &f_{zy}\eta + f_{zy'}\eta' + f_{zz}\zeta + f_{zz'}\zeta' - \frac{d}{dx}(f_{z'y}\eta + f_{z'y'}\eta' + f_{z'z}\zeta + f_{z'z'}\zeta') = 0, \end{aligned}$$

as may be shown by differentiating the two Euler equations with respect to u or v. The arguments of the derivatives of f are x, φ, φ', ψ, ψ'. The two equations (17) are linear and of the second order in η, ζ. It is therefore possible to express the solutions φ_u, ψ_u, and φ_v, ψ_v and hence the equation determining the focal point, in terms of a fundamental set of solutions of the Jacobi equations; this will be done in the next section.

§ 2. *The geometrical determination of the focal point.*

Since the Jacobi equations (17) are linear and of the second order, they have a fundamental system of solutions which with their derivatives may be denoted by

$$(18)\qquad \begin{matrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\ \eta_1' & \eta_2' & \eta_3' & \eta_4' \\ \zeta_1' & \zeta_2' & \zeta_3' & \zeta_4' \end{matrix}$$

and may be chosen so that the matrix of their values at $x = x_0$ is

$$(19)\qquad \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1. \end{matrix}$$

The particular solutions which occur in the equation

$$(20)\qquad \begin{vmatrix} \varphi_u & \psi_u \\ \varphi_v & \psi_v \end{vmatrix} = 0,$$

from which is determined the abscissa x of the point of contact of each of the extremals (10) with their enveloping surface, may be expressed in the form

$$\varphi_u = \sum_{i=1}^{4} c_i \eta_i, \qquad \psi_u = \sum_{i=1}^{4} c_i \zeta_i,$$
$$\varphi_v = \sum_{i=1}^{4} d_i \eta_i, \qquad \psi_v = \sum_{i=1}^{4} d_i \zeta_i. \tag{21}$$

From the initial conditions (19) it is evident that

$$c_1 = \varphi_u|^0, \quad c_2 = \psi_u|^0, \quad c_3 = \varphi'_u|^0, \quad c_4 = \psi'_u|^0,$$
$$d_1 = \varphi_v|^0, \quad d_2 = \psi_v|^0, \quad d_3 = \psi'_v|^0, \quad d_4 = \psi'_v|^0, \tag{22}$$

where the bar and superscript indicate that the arguments of the functions are for the point 0. At the intersection of the extremals with L, the equations (11) hold, and therefore by differentiating we have

$$y_u = \varphi' x_u + \varphi_u, \qquad z_u = \psi' x_u + \psi_u,$$
$$0 = \varphi_v, \qquad 0 = \psi_v, \tag{23}$$

in which the arguments of φ, ψ and their derivatives are $x(u), u, v$. Hence if u is taken as the length of arc L so that the derivatives x_u, y_u, z_u, are the direction cosines α, β, γ of the positive tangent to L, the constants (22) become

$$c_1 = \beta - \varphi'\alpha\,|^0, \quad c_2 = \gamma - \psi'\alpha\,|^0, \quad c_3 = \varphi'_u\,|^0, \quad c_4 = \psi'_u\,|^0,$$
$$d_1 = 0, \quad d_2 = 0, \quad d_3 = \varphi'_v\,|^0, \quad d_4 = \psi'_v\,|^0. \tag{24}$$

Since

$$d_1 = d_2 = 0,$$

the equation for the focal point, when (21) is substituted in (20), reduces to

$$c_1 d_3 (\eta_1 \zeta_3) + c_1 d_4 (\eta_1 \zeta_4) + c_2 d_3 (\eta_2 \zeta_3) + c_2 d_4 (\eta_2 \zeta_4) + (c_3 d_4)(\eta_3 \zeta_4) = 0, \tag{25}$$

the round brackets indicating second order determinants. The coefficients in this equation which involve c_3, c_4, d_3, d_4 can be calculated by means of the transversality condition

$$A = p x_u + q y_u + r z_u = 0,$$

in which the arguments of p, q, r are x, $\varphi[x(u), u, v,]$, $\varphi'[x(u), u, v]$ and the corresponding expressions for ψ. The equation is an identity in u, v since L is transversal to every curve of the family. If the u and v derivatives of (6) are found, we have

$$p x_{uu} + q y_{uu} + r z_{uu} + A' x_u + A_y \varphi_u + A_z \psi_u + A_{y'} \varphi'_u + A_{z'} \psi'_u = 0,$$
$$A_y \varphi_v + A_z \psi_v + A_{y'} \varphi'_v + A_{z'} \psi'_v = 0, \tag{26}$$

where the primes denote as always differentiation with respect to x. The derivatives of p, q, r with respect to x are

$$p' = f_x + f_y\varphi' + f_z\psi' - f'_{y'}\varphi' - f'_{z'}\psi', \qquad q' = f'_{y'}, \qquad r' = f'_{z'}$$

and since the curves (10) are all extremals satisfying the Euler equations (5), these reduce to

$$p' = f_x, \qquad q' = f_y, \qquad r' = f_z.$$

Hence the derivatives of A occurring in (26) have the values

$$\begin{aligned}
A' &= f_x\alpha + f_y\beta + f_z\gamma,\\
A_y &= (f_y - f_{y'y}\varphi' - f_{z'y}\psi')\,\alpha + f_{y'y}\beta + f_{z'y}\gamma,\\
A_z &= (f_z - f_{y'z}\varphi' - f_{z'z}\psi')\,\alpha + f_{y'z}\beta + f_{z'z}\gamma,\\
A_{y'} &= (\,-f_{y'y'}\varphi' - f_{z'y'}\psi')\,\alpha + f_{y'y'}\beta + f_{z'y'}\gamma = f_{y'y'}\varphi_u + f_{z'y'}\psi_u,\\
A_{z'} &= (\,-f_{y'z'}\varphi' - f_{z'z'}\psi')\,\alpha + f_{y'z'}\beta + f_{z'z'}\gamma = f_{y'z'}\varphi_u + f_{z'z'}\psi_u.
\end{aligned} \tag{27}$$

In (26) we will replace x_{uu}, y_{uu}, z_{uu} by their equals $l/\rho, m/\rho, n/\rho$, where the direction cosines l, m, n, are those of the principal normal to L at its intersection with the extremal (4), and ρ is the radius of curvature of L at the same point. Since equations (26) hold along L, and since φ_v, ψ_v vanish at 0, we have

$$d_3 : d_4 = -A_{z'} : A_{y'}.$$

In this ratio the expressions $A_{y'}$, and $A_{z'}$, cannot both vanish at the point 0. For if they did, it follows from their values given in (27) that φ_u and ψ_u would have to be zero, since the determinant of their coefficients is not zero. But in that case from equations (23) we should obtain the ratio

$$\alpha : \beta : \gamma = 1 : \varphi' : \psi',$$

and the transversality condition (6) would become

$$f - f_{y'}\varphi' - f_{z'}\psi' + f_{y'}\varphi' + f_{z'}\psi' = f = 0,$$

which contradicts the hypothesis of §1 that $f \neq 0$ at 0. Furthermore d_3, d_4 cannot both be zero, since the determinant (12) which is known to be different from zero near 0* would then vanish identically. The equation (25) is homogeneous in d_3, d_4, and hence the value of x which satisfies it is unchanged if we put

$$d_3 = -A_{z'}|^0, \quad d_4 = A_{y'}|^0. \tag{28}$$

From (22), (28) and the first equation of (26), it follows that

$$(c_3, d_4) = \varphi'_u A_{y'} + \psi'_u A_{z'} = -\left\{\frac{pl + qm + rn}{\rho} + A'x_u + A_y\varphi_u + A_z\psi_u\right\}.$$

* Mason and Bliss, loc. cit., p. 462.

If the values of A', A_y and A_z from (27) and of φ_u and ψ_u from (23) are substituted, the expression for (c_3, d_4) reduces to

$$(29) \qquad (c_3, d_4) = -\left\{ \frac{pl + qm + rn}{\rho} + Q(\alpha, \beta, \gamma) \right\},$$

where $Q(\alpha, \beta, \gamma)$ is the quadratic form

$$(30) \qquad \begin{aligned} Q(\alpha, \beta, \gamma) = &\{f_x - \varphi'(f_y - \varphi' f_{yy'} - \psi' f_{yz'}) - \psi'(f_z - \varphi' f_{yz'} - \psi' f_{zz'})\}\alpha^2 \\ &+ f_{yy'}\beta^2 + f_{zz'}\gamma^2 + \{2f_y - 2\varphi' f_{yy'} - \psi'(f_{yz'} + f_{y'z})\}\alpha\beta \\ &+ \{2f_z - \varphi'(f_{yz'} + f_{y'z}) - 2\psi' f_{zz'}\}\alpha\gamma + (f_{yz'} + f_{y'z})\beta\gamma, \end{aligned}$$

whose coefficients depend upon the values of f and its partial derivatives at the point 0 on C_{01}. The coefficients of the determinants (η_i, ζ_k) in (25) are therefore all expressible in terms of the directions of the principal normal and the curvature of L at 0, and of the directions of the extremal C_{01} to which L is transversal at that point.

The equation $(\eta_3, \zeta_4) = 0$ is satisfied by x_0, and its nearest root x_3 to x_0 defines the point "conjugate" to 0 on C_{01}, when J is to be minimized with respect to curves joining the point 0 with the fixed point 1. Unless the focal point for L and this conjugate point coincide on C_{01}, the determinant (η_3, ζ_4) is different from zero for the value of x which satisfies (25), and it may be written in the form

$$(31) \qquad \frac{(c_1\eta_1 + c_2\eta_2, d_3\zeta_3 + d_4\zeta_4)}{(\eta_3, \zeta_4)} = \frac{pl + qm + rn}{\rho} + Q(\alpha, \beta, \gamma).$$

The case where the focal and conjugate points coincide will be considered later. Equation (6) is the condition that the direction p, q, r is perpendicular to L and therefore lies in its normal plane. Hence the equation

$$(32) \qquad \frac{pl + qm + rn}{\sqrt{p^2 + q^2 + r^2}} = \cos\theta$$

defines θ as the angle between the principal normal of L and the normal p, q, r. Let the zero of equation (25) which is nearest to x_0 be denoted by x_2, and the segment on the line p, q, r which projects into ρ, the radius of curvature of L, by

$$(33) \qquad \pi = \rho \sec\theta,$$

as shown in the accompanying figure. Then the equation which x_2 must satisfy may be written

$$(34) \qquad H(x) = \frac{\sqrt{p^2 + q^2 + r^2}}{\pi} + Q(\alpha, \beta, \gamma),$$

where $H(x)$ is the left member of (31). From this equation it follows readily *that if a space curve L cuts an extremal C transversally and has a fixed direction at the intersection point 0, then the abscissa of the focal point of L on the extremal C depends only upon a quantity π, which if measured off on the normal p, q, r to L will project orthogonally into the radius of curvature of L at 0.* The functions p, q, r and their arguments are those defined by equations (7).

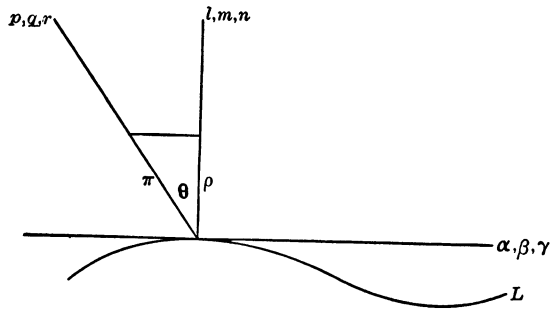

FIG. 1.

§ 3. *Conjugate solutions of Jacobi's equations.*

In order to discuss more completely the dependence of the focal point upon the projection π defined by equation (33), it will be necessary to use a formula due to VON ESCHERICH* involving so-called conjugate systems of solutions of the Jacobi equations. The method of deriving this formula as given here seems simpler for the present problem that that of VON ESCHERICH.

The definition of a conjugate system of solutions of the Jacobi equations involves an expression which will now be derived. It can readily be shown that when η and ζ are thought of as taking the place of y and z, the Jacobi equations are exactly the Euler equations of the homogeneous quadratic form

$$(35)\quad \begin{aligned}\Omega(\eta, \zeta) = f_{yy}\eta^2 + 2f_{yz}\eta\zeta + f_{zz}\zeta^2 + 2(f_{yy'}\eta\eta' + f_{yz'}\eta\zeta' + f_{zy'}\zeta\eta' + f_{zz'}\zeta\zeta') \\ + f_{y'y'}\eta'^2 + 2f_{y'z'}\eta'\zeta' + f_{z'z'}\zeta'^2,\end{aligned}$$

the reasons for this relationship appearing more clearly in § 5. Since Ω is homogeneous in $\eta, \eta', \zeta, \zeta'$ it has the following two useful properties:

$$(36)\qquad \Sigma(\eta\Omega_\eta + \eta'\Omega_{\eta'}) = 2\Omega,$$

$$(37)\qquad \Sigma(\eta\Omega_\zeta + \eta'\Omega_{\zeta'}) = \Sigma(\zeta\Omega_\eta + \zeta'\Omega_{\eta'}),$$

where the summations are taken with respect to η and ζ. Let the Euler

* VON ESCHERICH, *Die zweite Variation der einfachen Integrale*, Sitzungsberichte der kaiserlichen Akademie der Wissenschaften in Wien., vol. 107, Abth. IIa (1898), p. 1232.

expressions for Ω be denoted by

(38) $$M(\eta,\zeta)=\Omega_\eta-\Omega'_{\eta'},\quad N(\eta,\zeta)=\Omega_\zeta-\Omega'_{\zeta'}.$$

Then

$$\eta_1 M(\eta_2,\zeta_2)=\eta_1\Omega_{\eta_2}+\eta_1'\Omega_{\eta_2'}-\frac{d}{dx}\eta_1\Omega_{\eta_2'},$$

and it follows from (37) that

(39) $$\begin{aligned}\Sigma[\eta_1 M(\eta_2,\zeta_2)-\eta_2 M(\eta_1,\zeta_1)]&=\frac{d}{dx}\Sigma(\eta_2\Omega_{\eta_1'}-\eta_1\Omega_{\eta_2'})\\&=\frac{d}{dx}\Psi(\eta_1,\zeta_1;\eta_2,\zeta_2),\end{aligned}$$

the summation indicated being taken with respect to η and ζ in each case. The expression $\Psi(\eta_1,\zeta_1;\eta_2,\zeta_2)$ has the value

(40) $$\begin{aligned}\tfrac{1}{2}\Psi(\eta_1,\zeta_1;\eta_2,\zeta_2)&=(f_{yz'}-f_{y'z})(\eta_1\zeta_2-\eta_2\zeta_1)+\eta_2(f_{y'y'}\eta_1'+f_{y'z'}\zeta_1')\\&\quad+\zeta_2(f_{y'z'}\eta_1'+f_{z'z'}\zeta_1')-\eta_2'(f_{y'y'}\eta_1+f_{y'z'}\zeta_1)-\zeta_2'(f_{y'z'}\eta_1+f_{z'z'}\zeta_1).\end{aligned}$$

From (39) it is evident that for any two solutions of the Jacobi equations

(41) $$\Psi(\eta_1,\zeta_1;\eta_2,\zeta_2)=\text{constant}.$$

A conjugate system of solutions of the Jacobi equations is defined as a pair of solutions $\eta_1,\zeta_1;\ \eta_2,\zeta_2$, *which are linearly independent and which satisfy the condition*

(42) $$\Psi(\eta_1,\zeta_1;\eta_2,\zeta_2)=0\cdot$$

We will now take any three sets of solutions $\eta,\zeta;\ \eta_1,\zeta_1;\ \eta_2,\zeta_2$, and from them form the determinants

(43) $$\Delta=\begin{vmatrix}\eta & \eta_1\\ \zeta & \zeta_1\end{vmatrix},\qquad \Delta_1=\begin{vmatrix}\eta_1 & \eta_2\\ \zeta_1 & \zeta_2\end{vmatrix}.$$

The expression $\Delta_1\Delta'-\Delta\Delta_1'$, whose value is needed in the later discussion, can be put in the form

(44) $$\Delta_1\Delta'-\Delta\Delta_1'=\zeta_2\begin{vmatrix}\eta' & \eta_1' & \eta_2'\\ \eta & \eta_1 & \eta_2\\ \zeta & \zeta_1 & \zeta_2\end{vmatrix}-\eta_2\begin{vmatrix}\zeta & \zeta_1' & \zeta_2'\\ \eta & \eta_1 & \eta_2\\ \zeta & \zeta_1 & \zeta_2\end{vmatrix}=\zeta_2\chi(\eta,\zeta)-\eta_2\omega(\eta,\zeta),$$

where the primes denote derivatives with respect to x. On the hypothesis that $\eta_1,\zeta_1;\ \eta_2,\zeta_2$ form a conjugate system, it is possible to express the functions χ and ω in terms of $\Psi(\eta_1,\zeta_1;\eta,\zeta)$ and $\Psi(\eta_2,\zeta_2;\eta,\zeta)$ as defined in (40).

The equations

$$(45)\qquad \chi(\eta, \zeta) = 0, \qquad \omega(\eta, \zeta) = 0$$

and likewise

$$(46)\qquad \Psi(\eta_1, \zeta_1; \eta, \zeta) = \Psi_1 = 0, \qquad \Psi(\eta_2, \zeta_2; \eta, \zeta) = \Psi_2 = 0,$$

are linear in the first derivatives of $\eta, \eta_1, \eta_2, \zeta, \zeta_1, \zeta_2$. Moreover inspection shows that $\eta_1, \zeta_1; \eta_2, \zeta_2$ form a fundamental set of solutions for (45), and since they form a conjugate system they are likewise solutions of (46). Hence the two systems (45) and (46) are equivalent and the former may be expressed linearly in terms of the latter in the form

$$(47)\qquad \chi = \lambda_1\Psi_1 + \lambda_2\Psi_2, \qquad \omega = \mu_1\Psi_1 + \mu_2\Psi_2.$$

A comparison of the coefficients of η' and ζ' on both sides of these equations with the help of (40) gives four equations from which the following values of λ and μ can be determined.

$$(48)\qquad \begin{aligned} \lambda_1 &= -\frac{f_{y'z'}\eta_2 + f_{z'z'}\zeta_2}{D}, & \mu_1 &= \frac{f_{y'y'}\eta_2 + f_{y'z'}\zeta_2}{D}, \\ \lambda_2 &= \frac{f_{y'z'}\eta_1 + f_{z'z'}\zeta_1}{D}, & \mu_2 &= -\frac{f_{y'y'}\eta_1 + f_{y'z'}\zeta_1}{D}, \end{aligned}$$

where

$$D = \begin{vmatrix} f_{y'y'} & f_{y'z'} \\ f_{y'z'} & f_{z'z'} \end{vmatrix} \neq 0.$$

If we now assume further that the solutions η, ζ form with η_2, ζ_2 a conjugate system, but not with η_1, ζ_1, then by definition Ψ_2 vanishes, but Ψ_1 is different from zero and equations (47) take the simpler form

$$(49)\qquad \chi = \lambda_1\Psi_1, \qquad \omega = \mu_1\Psi_1.$$

Substitution of these values in (44) gives it the form

$$(50)\quad \Delta_1\Delta' - \Delta\Delta_1' = \Psi_1(\zeta_2\lambda_1 - \eta_2\mu_1) = -\frac{\Psi_1}{D}\{f_{y'y'}\eta_2^2 + 2f_{y'z'}\eta_2\zeta_2 + f_{z'z'}\zeta_2^2\}.$$

Hence if η_1, ζ_1 and η_2, ζ_2 form a conjugate system of solutions of the Jacobi equations, and if η, ζ is a third system conjugate to η_2, ζ_2 but not to η_1, ζ_1, then the determinants

$$\Delta = \begin{vmatrix} \eta & \eta_2 \\ \zeta & \zeta_2 \end{vmatrix}, \qquad \Delta_1 = \begin{vmatrix} \eta_1 & \eta_2 \\ \zeta_1 & \zeta_2 \end{vmatrix}$$

satisfy the relation (50).

§4. *Dependence of the focal point upon the curvature of the fixed curve L.*

The expression for $H(x)$, which is the left member of (31), may be written in the form

$$H(x) = d_4 \frac{(c_1\eta_1 + c_2\eta_2,\ d_3\zeta_3 + d_4\zeta_4)}{(\eta_3,\ d_3\zeta_3 + d_4\zeta_4)} = d_4 \frac{\Delta}{\Delta_1} \tag{51}$$

provided that d_4 is different from zero. In this expression the functions η and ζ have the meanings of § 2, while the three solutions (η, ζ), (η_1, ζ_1), (η_2, ζ_2) discussed in § 4 correspond to $(c_1\eta_1 + c_2\eta_2, c_1\zeta_1 + c_2\zeta_2)$, (η_2, ζ_3), $(d_3\eta_3 + d_4\eta_4, d_3\zeta_3 + d_4\zeta_4)$. The argument would not be changed materially if d_4 vanished, for since d_3 would then be different from zero, a similar transformation involving it could be made. The expressions $\Psi(\eta_i, \zeta_i;\ \eta_k, \zeta_k)$ are constants, since (η_i, ζ_i), (η_k, ζ_k) are solutions of the Jacobi equations for all of the values

$$i, k = 1, 2, 3, 4.$$

This constant, which will be denoted by Ψ_{ik}, can be calculated from (40) by putting $x = x_0$ and using the initial values (19), with the following results:

$$\begin{aligned} &\Psi_{12} = 2f_{yz'} - 2f_{y'z}\,|^0, && \Psi_{23} = -\,2f_{y'z'}\,|^0, \\ &\Psi_{13} = -\,2f_{y'y'}\,|^0, && \Psi_{24} = -\,2f_{z'z'}\,|^0, \\ &\Psi_{14} = -\,2f_{y'z'}\,|^0, && \Psi_{34} = 0. \end{aligned} \tag{52}$$

It follows directly that

$$\Psi(\eta_3, \zeta_3,\ d_3\eta_3 + d_4\eta_4,\ d_3\zeta_3 + d_4\zeta_4) = d_3\Psi_{33} + d_4\Psi_{34} = 0,$$

and therefore that the solutions in the denominator of $H(x)$ form a conjugate system. Similarly the solutions in the numerator turn out to be conjugate, for the function Ψ with these arguments reduces to zero by the aid of (22), (27), and (28). Furthermore since the expression

$$\Psi(c_1\eta_1 + c_2\eta_2, c_1\zeta_1 + c_2\zeta_2;\ \eta_3, \zeta_3) = 2\,\{c_1 f_{y'y'} + c_2 f_{y'z'}\}\,|^0 = 2d_4 \neq 0, \tag{53}$$

the solutions occurring in the expression (51) for $H(x)$ have exactly the properties discussed in § 3. Hence the values of $H'(x)$ can at once be written out by substituting for η_2, ζ_2 in (50) the corresponding solutions in the present case, and for Ψ_1 its value from equation (53). Then

$$\begin{aligned} H'(x) = d_4 \frac{\Delta_1\Delta' - \Delta\Delta_1'}{\Delta_1^2} = -\frac{2d_4^2}{D\Delta_1^2}\,\{&f_{y'y'}(d_3\eta_3 + d_4\eta_4)^2 \\ &+ 2f_{y'z'}(d_3\eta_3 + d_4\eta_4)(d_3\zeta_3 + d_4\zeta_4) + f_{z'z'}(d_3\zeta_3 + d_4\zeta_4)^2\}, \end{aligned} \tag{54}$$

which, on account of the properties of the quadratic form in the right member, is always negative for $x_0 < x < x_3$.

Both numerator and denominator of $H(x)$ vanish at x_0, but the expansion of the functions η_i, ζ_i involved in the neighborhood of x_0 by Taylor's formula with a remainder term determines the value of $H(x)$ as a finite quantity. On account of the initial conditions (19), the first terms of these expansions are as follows:

$$\eta_1 = 1 + \cdots, \qquad \eta_2 = \alpha_2 (x - x_0)^2 + \cdots, \qquad \eta_3 = (x - x_0) + \cdots,$$
$$\eta_4 = \alpha_4 (x - x_0)^2 + \cdots, \qquad \zeta_1 = \beta_1 (x - x_0)^2 + \cdots,$$
$$\zeta_2 = 1 + \cdots, \qquad \zeta_3 = \beta_3 (x - x_0)^2 + \cdots, \qquad \zeta_4 = (x - x_0) + \cdots.$$

The first terms of the corresponding expansions for Δ and Δ_1 will therefore be

$$\Delta = (c_1 d_4 - c_2 d_3)(x - x_0) + \cdots,$$
$$\Delta_1 = d_4 (x - x_0)^2 + \cdots.$$

The coefficient of $x - x_0$ in Δ reduces by (28) to the quadratic form

$$c_1^2 f_{y'y'} + 2c_1 c_2 f_{y'z'} + c_2^2 f_{z'z'} \,|^0,$$

which is positive. Hence $H(x)$ approaches the value $+\infty$ as $x - x_0$ approaches zero through positive values.

The function $H(x)$ also becomes infinite as x approaches the value x_3 defining the point conjugate to x_0, for the determinant Δ_1 vanishes at x_3 and in the present case it has been assumed that Δ does not. Furthermore since its derivative (54) does not change sign, $H(x)$ varies monotonically.

Hence as x varies from x_0 to x_3 the function $H(x)$ takes every real value once and only once, and therefore for any value of π the equation (34) *has a unique root. In other words any curve transversal to C_{01} at* 0, *and having the same tangent with L at* 0, *has a unique focal point on C between the intersection point* 0 *and its conjugate* 3.

To determine how the focal point 2 varies with π when the extremal C_{01} and the direction of L remain fixed while the curvature of L at 0 is allowed to vary, the derivative of x_2 with respect to π must be found from (34).

The resulting expression,

$$\frac{dx_2}{d\pi} = -\frac{\sqrt{p^2 + q^2 + r^2}}{\pi^2 H'(x_2)} \tag{55}$$

is positive for $x_0 < x < x_3$. Further, since $H(x)$ is infinite at x_0 and x_3, equation (34) shows that these values must correspond to $\pi = 0$. *It follows that as the projection π varies from* 0 *to* $+\infty$ *and from* $-\infty$ *to* 0, *the focal point traverses the curve C monotonically from* 0 *to* 3.

In case the point 0 has no conjugate on the curve C a value of π can be selected so near to zero that equation (34) has a corresponding root x_2. By the usual theorems on implicit functions it follows that the equations can be solved for x_2 as a function $x_2(\pi)$ in the neighborhood of this initial solution, and the function so determined can be continued indefinitely unless a limiting value x_2 is approached at which the continuity properties of the functions entering into the equation no longer hold. The value of $x_2(\pi)$ will increase monotonically as π increases. *In other words, if there is no point* 3 *conjugate to* 0 *on* C, *then as the value of* π *increases, the focal point* 2 *of* L *will traverse the curve* C *monotonically until it approaches the end point of the arc* C *at which the properties prescribed for the function* f *cease to hold or else* 2 *will move off to infinity on* C.

In the exceptional case where Δ and Δ_1 vanish simultaneously at x_3, the expression $\Delta_1\Delta' - \Delta\Delta_1'$ will also vanish. Inspection of equation (54) shows that this can happen only when

$$(56) \qquad d_3\eta_3 + d_4\eta_4 = 0, \qquad d_3\zeta_3 + d_4\zeta_4 = 0,$$

on account of the properties of the quadratic form which it involves. If the values of d_3 and d_4 from (28) are substituted in (56), two equations linear in α, β, γ are obtained:

$$(57) \quad \begin{aligned} &\{ (f_{y'z'}\varphi' + f_{z'z'}\psi')\,\eta_3 - (f_{y'y'}\varphi' + f_{y'z'}\psi')\,\eta_4 \}\,\alpha \\ &\qquad - \{ f_{y'z'}\eta_3 - f_{y'y'}\eta_4 \}\,\beta - \{ f_{z'z'}\eta_3 - f_{y'z'}\eta_4 \}\,\gamma = 0, \\ &\{ (f_{y'z'}\psi' + f_{z'z'}\varphi')\,\zeta_3 - (f_{y'y'}\varphi' + f_{y'z'}\psi')\,\zeta_4 \}\,\alpha \\ &\qquad - \{ f_{y'z'}\zeta_3 - f_{y'y'}\zeta_4 \}\,\beta - \{ f_{z'z'}\zeta_3 - f_{y'z'}\zeta_4 \}\,\gamma = 0. \end{aligned}$$

These equations are not independent, since all the determinants of their matrix vanish. However, either one is a condition on α, β, γ, independent of the transversality condition (6), since it turns out that the determinants of the corresponding matrix cannot vanish unless

$$\eta_3 = \eta_4 = \zeta_3 = \zeta_4.$$

Except in this last case, either one of equations (57) together with the transversality condition (6) will determine the ratio of α, β, γ uniquely, *and there is therefore only one direction,* α, β, γ, *transversal to* C_{01} *at* 0 *for which* Δ *and* Δ_1 *vanish simultaneously at the conjugate point.*

If Δ_1 has a zero of higher order than Δ at x_3, $H(x_3)$ is infinite; and it follows from equation (34) that x_2 will approach x_3 only when π approaches zero through negative values, and the focal point varies with π as in the previous case.. But if the zero of Δ_1 is of the same order as Δ or lower, then $H(x_3)$ is finite and x_2 approaches x_3 as π approaches a certain value π_3 different from zero; that is, *the focal point* 2 *traverses* C *from* 0 *to* 3 *while* π *increases from zero to a value* π_3, *zero or infinite; and as* π *increases from* π_3 *through negative values to* 0, *the focal point will remain fixed at* x_3. For since the point 3 is conjugate to 0, no arc containing the point 3 can minimize J with respect to curves joining L and 1.

§ 5. *The second variation when one end point is variable.*

The necessary condition that C_{01} minimize the integral J can be derived from those of an ordinary mimimum problem by considering a one-parameter family of curves

$$y = g(x, u), \qquad z = h(x, u), \tag{58}$$

which include the arc C for $u = 0$, pass through the point 1 when $x = x_1$ and intersect the curve L. Analytically these properties of the family (58) are expressed by the equations

$$\begin{aligned} \varphi(x) &= g(x, 0), & \psi(x) &= h(x, 0), \\ y_1 &= g(x_1, u), & z_1 &= h(x_1, u), \\ y(u) &= g[x(u), u], & z(u) &= h[x(u), u]. \end{aligned} \tag{59}$$

The integral

$$J(u) = \int_{x(u)}^{x_1} f(x, g, g' h, h')\,dx \tag{60}$$

taken along any curve of the family is a function of u which is to be minimized for $u = 0$. Therefore the following conditions must be satisfied:

$$J'(0) = 0, \qquad J''(0) \geqq 0.$$

On differentiating (60) the derivative $J'(u)$ is found to have the value

$$J'(u) = f x_u |_{x(u)} + \int_{x(u)}^{x_1} \Sigma\,(f_y g_u + f_{y'} g'_u)\,dx, \tag{61}$$

where the sum is to be taken with respect to y, z or corresponding symbols such as g, h. It appears after an integration by parts that $J'(0)$ vanishes only if

$$f x_u + f_{y'} g_u + f_{z'} h_u = 0 \tag{62}$$

at $x = x_0$, the intersection of L and C_{01}. This becomes the transversality condition (6) when g_u and h_u are replaced by their values

$$g_u = y_u - g' x_u, \qquad h_u = z_u - h' x_u, \tag{63}$$

found from equations (59).

Differentiating (61) again and evaluating for $u = 0$, we have

$$\begin{aligned} J''(0) = {} & f x_{uu} + f_u x_u + \Sigma(f_y \eta + f_{y'} \eta')|_0 \\ & + \int_{x_0}^{x_1} \Sigma(f_y g_{uu} + f_{y'} g'_{uu})\,dx + \int_{x_0}^{x_1} \Omega(\eta, \zeta)\,dx, \end{aligned} \tag{64}$$

where

$$\eta(x) = g_u(x, 0), \qquad \zeta(x) = h_u(x, 0), \tag{65}$$

and Ω has the value given in equation (35).

For $x = x_0$, the values of $\eta(x)$ and $\zeta(x)$ are given by equations (63), and from (59) g_{uu}, h_{uu} satisfy

$$\begin{aligned} y_{uu} &= g' x_{uu} + g'' x_u^2 + 2g_u' x_u + g_{uu}, \\ z_{uu} &= h' x_{uu} + h'' x_u^2 + 2h_u' x_u + h_{uu}, \end{aligned} \tag{66}$$

while at $x = x_1$ all four of these functions vanish. Hence if the usual integration by parts be applied to the first integral of (64), since C_{01} is an extremal the expression $J''(0)$ becomes

$$\begin{aligned} J''(0) = {} & p x_{uu} + q y_{uu} + r z_{uu} + (f_x - f_y \varphi' - f_z \psi') x_u^2 \\ & + 2 x_u y_u f_y + 2 x_u z_u f_z \Big|_0 + \int_{x_0}^{x_1} \Omega(\eta, \zeta)\, dx. \end{aligned} \tag{67}$$

The expression outside of the integral sign, it is to be noted, is independent of the choice of η and ζ. From equation (36) it follows that the integrand in the last equation can be put in the form

$$\tfrac{1}{2} \Sigma (\Omega_\eta \eta + \Omega_{\eta'} \eta') \tag{68}$$

and integrated by parts. On substituting the derivatives of Ω and again applying equation (63) to the terms outside the integral sign, at the same time substituting for x_u, y_u, z_u, their equals α, β, γ, the expression (67) takes the form

$$\begin{aligned} J''(0) = {} & p x_{uu} + q y_{uu} + r z_{uu} + Q(\alpha, \beta, \gamma) + \{ (-f_{y'y'} \varphi' - f_{y'z'} \psi') \alpha \\ & + f_{y'y'} \beta + f_{y'z'} \gamma \} \eta' + \{ (-f_{y'z'} \varphi' - f_{z'z'} \psi') \alpha + f_{y'z'} \beta + f_{z'z'} \gamma \} \zeta' \Big|_0 \\ & + \tfrac{1}{2} \int_{x_0}^{x_1} \Sigma \eta (\Omega_\eta - \Omega_{\eta'}') \, dx, \end{aligned} \tag{69}$$

where $Q(\alpha, \beta, \gamma)$ is the same quadratic form as that given by equation (30), and the expression $\Omega_\eta - \Omega_{\eta'}'$ and $\Omega_\zeta - \Omega_{\zeta'}'$ under the integral sign when equated to zero,

$$\begin{aligned} \Omega_\eta - \Omega_{\eta'}' &= 0, \\ \Omega_\zeta - \Omega_{\zeta'}' &= 0, \end{aligned} \tag{70}$$

are the Euler equations (5) for the function $\Omega(x, \eta, \eta', \zeta, \zeta')$, as well as the Jacobi equations (17).

It will now be shown that under certain circumstances the family of curves (58) can be so selected that $J''(0)$ will vanish, and that $J''(0)$ may even be made to take opposite signs, a condition which cannot exist if C_{01} is to render J a minimum.

First let us consider the conditions under which $\eta(x)$ and $\zeta(x)$, functions of class C', may be the derivatives $g_u(x, 0)$, $h_u(x, 0)$ for a family of variations (58). From equations (59) it follows that

$$\eta(x_1) = g_u(x_1, 0) = 0, \qquad \zeta(x_1) = h_u(x_1, 0) = 0,$$
$$\eta(x_0) = y_u - \varphi'(x_0)x_u, \qquad \zeta(x_0) = z_u - \psi'(x_0)x_u. \tag{71}$$

These equations are the necessary and sufficient conditions that a family of variations (58) *exists with* $\eta(x)$, $\zeta(x)$ *equal to the derivatives* $g_u(x, 0)$, $h_u(x, 0)$. For let $\eta(x)$, $\zeta(x)$ be two equations satisfying these conditions and let

$$y = \varphi(x) + u\eta(x) + (x - x_1)Y(u),$$
$$z = \psi(x) + u\zeta(x) + (x - x_1)Z(u) \tag{72}$$

be a particular form of (58), where $Y(u)$, $Z(u)$ are to be determined. Evidently the curves of the family all pass through the point 1. Furthermore at the intersection of the family with L, $Y(u)$, $Z(u)$ can be determined so that

$$y(u) = \varphi[x(u)] + u\eta[x(u)] + [x(u) - x_1]Y(u),$$
$$z(u) = \psi[x(u)] + u\zeta[x(u)] + [x(u) - x_1]Z(u). \tag{73}$$

For $u = 0$ it follows, since L and C_{01} intersect at 0, that

$$Y(0) = Z(0) = 0. \tag{74}$$

Differentiation of the first of equation (73) gives

$$y_u = \varphi' x_u + \eta + u\eta' x_u + x_u Y(u) + [x(u) - x_1]Y_u. \tag{75}$$

On account of equations (71) and (74), the evaluation of (75) and of the analogous equation for z when $u = 0$ give

$$Y_u(0) = Z_u(0) = 0.$$

Therefore $Y(u)$ and $Z(u)$ have the form $u^2Y_1(u)$, $u^2Z_1(u)$ where Y_1 and Z_1 are of class C' near $u = 0$. Equations (72) now take the form

$$y = \varphi(x) + u\eta(x) + (x - x_1)u^2Y_1(u) = g(x, u),$$
$$z = \psi(x) + u\zeta(x) + (x - x_1)u^2Z_1(u) = h(x, u), \tag{76}$$

and it is evident on differentiation that $\eta(x)$, $\zeta(x)$ are equal to $g_u(x, 0)$, $h_u(x, 0)$ respectively.

It is now possible to determine conditions under which η and ζ may be selected so that equations (59) are satisfied and also so that $J''(0)$ vanishes. If η, ζ is a solution of the Jacobi equations (17) or (70), each can be expressed in terms of the fundamental set (18), and they may be taken in the form

(77) $$\eta = \sum_{i=1}^{4} c_1 \eta_i, \qquad \zeta = \sum_{i=1}^{4} c_i \zeta_i,$$

where the constants are to be determined by conditions (19) and (71). It is at once evident that

(78) $$\begin{aligned} c_1 &= \eta\,|^{00} = y_u - \varphi'(x_0)\,x_u, & c_2 &= \zeta\,|^{00} = z_u - \psi'(x_0)\,x_u, \\ c_3 &= \eta'\,|, & c_4 &= \zeta'\,|. \end{aligned}$$

If equations (71) are to be satisfied and at the same time the expression (69) for $J''(0)$ is to vanish, c_3 and c_4 must satisfy the three equations

(79) $$\begin{gathered} c_1\eta_1(x_1) + c_2\eta_2(x_1) + c_3\eta_3(x_1) + c_4\eta_4(x_1) = 0, \\ c_1\zeta_1(x_1) + c_2\zeta_2(x_1) + c_3\zeta_3(x_1) + c_4\zeta_4(x_1) = 0, \\ U + c_3 V + c_4 W\,|_0 = 0, \end{gathered}$$

where

$$\begin{aligned} U &= p x_{uu} + q y_{uu} + r z_{uu} + Q(a, B, \gamma), \\ V &= (-\varphi' f_{y'y'} - \psi' f_{y'z'})\,\alpha + f_{y'y'}\beta + f_{y'z'}\gamma, \\ W &= (\varphi' f_{y'z'} - \psi' f_{z'z'})\,\alpha + f_{y'z'}\beta + f_{z'z'}\gamma. \end{aligned}$$

If these can be solved for c_3 and c_4 the functions η and ζ, which will then be completely determined, will make $J''(0)$ vanish.

If the determinant

(80) $$\begin{vmatrix} c_1\eta_1(x) + c_2\eta_2(x) & \eta_3(x) & \eta_4(x) \\ c_1\zeta_1(x) + c_2\zeta_2(x) & \zeta_3(x) & \zeta_4(x) \\ U & V & W \end{vmatrix}$$

vanishes for any value x_2 between x_0 and x_4, then η and ζ can be so chosen that $J''(0)$ will be either positive or negative, and consequently the arc C_{01} cannot minimize the integral J.

In order to show this, suppose first that at x_2 the condition

(81) $$\begin{vmatrix} \eta_3(x) & \eta_4(x) \\ \zeta_3(x) & \zeta_4(x) \end{vmatrix} \neq 0$$

is satisfied. Then the two equations which differ from the first two of (79) only in having the argument x_1 replaced by x_2, can be solved for c_3 and c_4, and their values, by the hypothesis that the determinant (80) is zero at x_2, satisfy the third equation of (79). Two functions $\eta(x)$, $\zeta(x)$ can therefore be chosen which are solutions of the Jacobi equations, have the values (71) at x_0, satisfy the equations analogous to (79) at x_2, and vanish identically between x_2 and x_1. For such a pair of functions $J''(0)$ is zero. The functions η, ζ so determined cannot be identically zero between x_0 and x_2. For if that were the case the expressions (71) would vanish at $x = x_0$, and by the same argument as that used in §2, this can be shown to be impossible, since $f \neq 0$ at 0. Hence at the point 2 at least one of the derivatives η', ζ' is not zero, since the only solutions of the Jacobi equations which can vanish simultaneously with their derivatives are $\eta \equiv \zeta \equiv 0$.

Consider now the problem of minimizing the integral

$$\int_{x_0}^{x_1} \Omega(\eta, \zeta)\, dx \tag{82}$$

with respect to curves satisfying the initial conditions (71). The functions $\eta(x)$, $\zeta(x)$ as chosen above are extremals for this integral since they are solutions of equations (70). They have however a corner point at x_2, and hence, if they are to minimize (82), the equations*

$$\Omega_{\eta'}^{-} = \Omega_{\eta'}^{+}, \qquad \Omega_{\zeta'}^{-} = \Omega_{\zeta'}^{+}, \tag{83}$$

must be satisfied at $x = x_2$. After these expressions are evaluated, since η, ζ and their right hand derivatives are zero at x_2, (83) reduces to

$$\begin{aligned} f_{y'y'}\eta'_{-} + f_{y'z'}\zeta'_{-} &= 0, \\ f_{y'z'}\eta'_{-} + f_{z'z'}\zeta'_{-} &= 0. \end{aligned} \tag{84}$$

These last equations cannot be satisfied, since their determinant is different from zero by (2) and it has just been proved that η'_{-}, ζ'_{-} do not both vanish at $x = x_2$. Therefore η and ζ as chosen do not minimize the integral when there is a corner point, and it is possible to choose two functions $\bar{\eta}$, $\bar{\zeta}$ which will give the integral (82) a smaller or a greater value than that given by the functions described above. Under these circumstances the expression (67) for $J''(0)$ can be made either negative or positive, since for $\eta(x)$, $\zeta(x)$ it vanishes, and since the terms outside the integral are independent of η and ζ.

The proof that C_{01} cannot minimize the integral J if a zero x_2 of the determinant (80) lies between x_0 and x_1, is now complete for the case when the expression (81) is not zero at x_2. If this last determinant does vanish at $x = x_2$, then two functions

$$\eta = c_3\eta_3(x) + c_4\eta_4(x), \qquad \zeta = c_3\zeta_3(x) + c_4\zeta_4(x) \tag{85}$$

can be chosen, which on account of (19) are zero at x_0 as well as at x_2, and which are identically zero between x_2 and x_1. The family of curves

$$y = \varphi(x) + u\,\eta(x), \qquad z = \psi(x) + u\,\zeta(x) \tag{86}$$

* Bolza, *Vorlesungen über Variationsrechnung*, p. 366.

all pass through the points 0 and 1. For these variations, all of which have the same end points, the expression for $J''(0)$ reduces simply to the integral (82). By the same argument as has just been used, it follows that $J''(0)$ vanishes for the η, ζ just defined, and that these functions cannot minimize $J''(0)$, since they have a corner point at x_2 at which the necessary conditions for a minimum are not satisfied.

By comparing (80) with equations (29) and (27) it is evident that the elements of the last row of the determinant are $-(c_3, d_4)$, d_4, and $-d_3$ respectively. Therefore if (80) is expanded in terms of these elements, the result when equated to zero is precisely the equation (25) whose solution x_2 determines the focal point on C_{01}. This identifies the focal point with the value x which is the zero of the determinant (80) in the foregoing discussion, and completes the proof, *made independently of the form of the surface enveloping the extremals, that if the arc C_{01} is to minimize the integral J with respect to curves joining L with the point* 1 *the focal point cannot lie between* 0 *and* 1.

We will now consider the case where the determinant (80) does not vanish for any value of x between x_0 and x_1. Since its zero determines the abscissa of the focal point 2 of L on the extremal, 2 does not lie on the arc C_{01}. It is proposed to show that in this case $J''(0)$ is positive for all choices of η and ζ, that is, for any one-parameter family of variations of the form (58). It will be shown that a proper choice of a special pair of functions η_0, ζ_0 satisfying conditions (71) will make the value $J''(0)$ positive and at the same time less than the corresponding value for any other pair η_1, ζ_1 satisfying (71).

Let η and ζ be in the form

$$\begin{aligned}\eta &= c_1\eta_1(x) + c_2\eta_2(x) + a\eta_3(x) + b\eta_4(x),\\ \zeta &= c_1\zeta_1(x) + c_2\zeta_2(x) + a\zeta_3(x) + b\zeta_4(x),\end{aligned} \tag{87}$$

which represent in (x, η, ζ)-space the four-parameter family of extremals of the integral in the formula (67) for $J''(0)$, and let η_0, ζ_0 be the particular curve that passes through 0 and 1. The second condition of (71) determines

$$c_1 = \beta - \varphi'\alpha\,|^0, \qquad c_2 = \gamma - \psi'\alpha\,|^0, \tag{88}$$

and we may determine the values a_0 and b_0 for the functions η_0, ζ_0 by means of the equations,

$$\begin{aligned}c_1\eta_1(x_1) + c_2\eta_2(x_2) + a_0\eta_3(x_1) + b_0\eta_4(x_1) &= 0,\\ c_1\zeta_1(x) + c_2\zeta_2(x_2) + a_0\zeta_3(x_1) + b_0\zeta_4(x_1) &= 0.\end{aligned} \tag{89}$$

The determinant of those equations with respect to a and b is different from zero, since 1 is by hypothesis between 0 and its conjugate point on C_{01}. If the values of c_1 and c_2 in equations (87) are determined by means of equations (88), while those of a and b are left arbitrary, a two-parameter family of extremals for the integral

$$\int_0^1 \Omega(\eta, \zeta)\, dx$$

(90) $$\eta = \eta(x, a, b), \qquad \zeta = \zeta(x, a, b)$$

is determined, all of which satisfy the second of the conditions (71). We will now show that the particular curve

$$\eta_0 = \eta(x, a_0, b_0), \qquad \zeta_0 = \zeta(x, a_0\, b_0)$$

satisfies conditions sufficient to minimize the integral in $J''(0)$ and furthermore makes $J''(0)$ positive.

By referring to the expressions for Ω, it is seen that the conditions

$$\Omega_{\eta'\eta'} > 0, \qquad \Omega_{\eta'\eta'}\Omega_{\zeta'\zeta'} - \Omega_{\eta'\zeta'} > 0$$

reduce to the corresponding conditions on the function f. Again, the determinant

$$\begin{vmatrix} \eta_a & \zeta_a \\ \eta_b & \zeta_b \end{vmatrix}$$

whose zero determines the focal point on the extremal, reduces to the determinant (η_3, ζ_4) which is different from zero for $x_0 < x \overline{\gtrless} x_3$, and therefore the focal point does not lie on the extremal in question. The E-function which has the form

$$E(x, \eta, \zeta, \eta', \zeta', \eta_0', \zeta_0') = \Omega(x, \eta, \zeta, \eta', \zeta') - \Omega(x, \eta, \zeta, \eta_0', \zeta_0') \\ - (\eta' - \eta_0')\,\Omega_{\eta_0'} - (\zeta' - \zeta_0')\,\Omega_{\zeta_0'}$$

can, by using Taylor's Theorem, be transformed into

$$\tfrac{1}{2}\left[\Omega_{\eta'\eta'}(\eta' - \eta_0')^2 + 2\Omega_{\eta'\zeta'}(\eta' - \eta_0')(\zeta' - \zeta_0') + \Omega_{\zeta'\zeta'}(\zeta' - \zeta_0')^2\right]$$

where the second derivatives of Ω are exactly the corresponding derivatives of f with respect to y, z instead of η, ζ, and where η', ζ' give the directions of any curve joining 0 and 1. This expression for the E-function of Ω is the same as that for the function f along the extremal C_{01} and is therefore positive. It follows that the particular functions η_0, ζ_0 minimize the integral $\int \Omega(\eta, \zeta)\, dx$ with respect to any other functions η and ζ satisfying the relations (71). These same functions also minimize $J''(0)$, since in its value given in (67) η and ζ enter only in the integrand.

It can also be shown that $J''(0)$ is positive for η_0 and ζ_0. The expression

(69) can be put in the form,

$$J''(0) = -\frac{\sqrt{p^2+q^2+r^2}}{\pi} - Q(\alpha, \beta, \gamma) - d_4\eta' + d_3\zeta' \Big|^0 + \frac{1}{2}\int_0^1 \Sigma(\Omega_\eta - \Omega'_{\eta'})\,\eta\,dx. \tag{91}$$

Since $\eta'(x_0)$ and $\zeta'(x_0)$ equal a_0 and b_0 respectively, if their values from (89) are substituted in (91) it becomes

$$J''(0) = \left\{ H(x_1) - \frac{\sqrt{p^2+q^2+r^2}}{\pi} - Q(\alpha, \beta, \gamma) \right\} \Bigg|^0 + \frac{1}{2}\int_0^1 \Sigma(\Omega_\eta - \Omega'_{\eta'})\,\eta\,dx. \tag{92}$$

It has been proved that $H(x_0) = +\infty$, and that the first value of x for which

$$H(x) - \frac{\sqrt{p^2+q^2+r^2}}{\pi} - Q(\alpha, \beta, \gamma)$$

becomes zero is x_2, the abscissa of the focal point, which in the present case does not lie on the arc C_{01}. Hence the expression in the parenthesis is positive, and as the choice of η and ζ causes the integral to vanish, $J''(0)$ is positive. Since a pair of functions η_0 and ζ_0 has been found which not only make $J''(0)$ positive, but also minimize it with respect to all arbitrarily chosen η, ζ satisfying conditions (71), we see that $J''(0)$ *is positive for any family (58) when the focal point does not lie on the minimizing arc, and* $J(C_{01})$ *is less than the value of* J *taken along any neighboring curve of the family.*

§6. *The focal point for a surface.*

The determination of the focal point on the extremal C_{01} which is to minimize the integral J with respect to curves joining a surface S and a fixed point 1, follows readily from the results for the curve. The necessary conditions which C_{01} must satisfy have been stated in §1. At the intersection of the surface S (14) with the two parameter family of extremals in which C_{01} is imbedded, the following equations hold:

$$y(u, v) = \varphi[x(u, v), u, v], \qquad z(u, v) = \psi[x(u, v), u, v], \tag{93}$$

and therefore

$$\begin{aligned} y_u &= \varphi' x_u + \varphi_u, & z_u &= \psi' x_u + \varphi_u, \\ y_v &= \varphi' x_v + \varphi_v, & z_v &= \psi' x_v + \psi_v.* \end{aligned} \tag{94}$$

* Eisenhart, *Differential Geometry*, p. 118.

By means of these equations, the transversality conditions (15) can be transformed into

$$(95) \qquad px_u + qy_u + rz_u = 0, \qquad px_v + qy_v + rz_v = 0,$$

from which it appears that the direction p, q, r is normal to the surface. By Meusnier's theorem, the radius of curvature at a point 0 of any curve L on the surface is the orthogonal projection of the radius of the normal section through 0 determined by the tangent to the original curve. The curve L is transversal to the extremal C, and, as has just been seen, the normal p, q, r is also the normal to the surface. *It follows that all curves on the surface that have a common tangent at* 0 *have the same focal point on the extremal* C.

To find the focal point of the surface, it will be sufficient to consider the focal points determined by normal sections of the surface. One of these will be the nearest to 0 on C and that one is also the focal point of the surface.

Let M be the normal section whose focal point m on C_{01} lies nearest 0. Suppose the focal point s of the surface lies beyond m, and select a point 3 between m and s. Then since m is the focal point for M, a curve V_{43} can be drawn intersecting M at 4 and such that

$$J(V_{43}) < J(C_{03}).$$

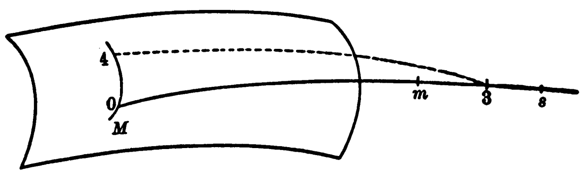

FIG. 2.

On the other hand, since 3 lies between 0 and s we know that the arc C_{03} must minimize J between s and 3; hence

$$J(C_{03}) < J(V_{43}),$$

which contradicts the former result. Therefore s cannot lie beyond m.

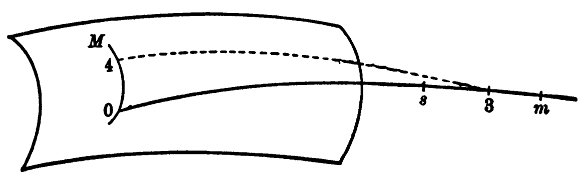

FIG. 3.

If s lies between 0 and m, we will choose as before a point 3 between s and m. Then as 3 lies beyond s, the extremal ceases to minimize J taken between the surface and the point 3, and a curve V_{34} can be drawn to the surface such that

$$J(V_{43}) < J(C_{03}). \tag{96}$$

In fact a family of variations V_{43} depending upon a parameter t, joining S to the point 3, and including C_{03} for $t = 0$, can be found, for each of which the last inequality holds. The coördinates of the intersection point 4 of the surface with one of these variations are determined by equations of the form

$$u_4 = u_4(t), \qquad v_4 = v_4(t),$$

which define a curve L on S passing through the point 0. Again, however, a contradiction arises, for the focal point of the curve L on C is the same as that of the normal section of S determined by the tangent to L, and therefore coincides with or lies beyond m. Consequently from the last results of § 5 the arc C_{03} minimizes J with respect to the curves of the family V_{43}, and the inequality (96) cannot be true.

We have proved therefore that *the focal point of the surface on the extremal* C_{01} *must coincide with that one of the focal points of the normal sections at* 0 *that is nearest to* 0 *on the extremal.*

THE UNIVERSITY OF CHICAGO,
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